On lattice-valued frames

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Pointfree topology

Motivation

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

$f \downarrow \qquad \uparrow f^{-1}$

$$(Y, \mathcal{O}Y) \rightsquigarrow (\mathcal{O}Y, \subseteq)$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

f^{-1} preserves $\bigcup$ and $\cap$

TOPOLOGY $\xrightarrow{\text{Abstraction}}$ POINTFREE TOPOLOGY

Pointfree topology the category of frames **Frm**

- The objects in **Frm** are frames, i.e.
  - $*$ complete lattices L in which
  - $*$ $a \wedge \bigvee_{i \in I} a_i = \bigvee \{a \wedge a_i : i \in I\}$ for all $a \in L$ and $\{a_i : i \in I\} \subseteq L$.
- Morphisms, called frame homomorphisms, are those maps between frames that preserve arbitrary joins and finite meets.
- $\mathcal{O}$: **Top** $\to$ **Frm** is a contravariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X$.

Pointfree topology the dual category $\mathbf{Loc} = \mathbf{Frm}^{op}$

- The objects in **Loc** are frames, from now on, also called locales.
- Morphisms, called localic maps, are of course, just frame homomorphisms taken backwards.
- $\mathcal{O} : \mathbf{Top} \to \mathbf{Loc}$ is now a covariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}X \xrightarrow{f^{-1}} \mathcal{O}Y$.

Advantage: **Loc** can be thought of as a natural extension of (sober) spaces.

Disadvantage: Morphisms thought in this way may obscure the intuition.

Pointfree Topology, Pointless Topology, Frame Theory, Locale Theory...

"*Locales not only "capture" or "model" the lattice theoretical behaviour of topological spaces, more importantly when we work in a universe where choice principles are not allowed, it is locales, not spaces, which provide the right context in which to do topology.*"

P.T. Johnstone, *The point of pointless topology*, Bull. Amer. Math. Soc. 8 (1983) 41-53.

Pointfree topology — spatial frames and sober spaces

Apart from the functor $\mathcal{O}$: **Top** $\to$ **Frm**, there is a functor in the opposite direction, the spectrum functor

$$\mathrm{Spec} : \mathbf{Frm} \to \mathbf{Top}$$

An element $p \in L \setminus \{1\}$ is called prime if for each $\alpha, \beta \in L$ with

$$\alpha \wedge \beta \leq p \quad \Longrightarrow \quad \alpha \leq p \text{ or } \beta \leq p.$$

We denote by $\mathrm{Spec}\, L$ the spectrum of L, i.e. the set of all prime elements of L.

The functor Spec assigns to each frame L its spectrum $\mathrm{Spec}\, L$, endowed with the hull-kernel topology whose open sets are

$$\Delta_L(\alpha) = \{p \in \mathrm{Spec}\, L : \alpha \not\leq p\} = \mathrm{Spec}\, L \setminus {\uparrow}\alpha \quad \text{for } \alpha \in L.$$

Pointfree topology spatial frames and sober spaces

We have an adjoint situation:

$$\textbf{Top} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \textbf{Frm}$$

Recall that a topological space X is sober if the only prime opens are those of the form $X \setminus \overline{\{x\}}$ for some $x \in X$ and a frame L is spatial if L is generated by its prime elements, i.e. if

$$\alpha = \bigwedge \{p \in \mathrm{Spec}\, L : \alpha \leq p\} = \bigwedge (\uparrow\alpha \cap \mathrm{Spec}\, L) \quad \text{for all } \alpha \in L,$$

The categories **Sob** of sober topological spaces and **SpatFrm** of spatial frames are dual under the restrictions of the functors $\mathcal{O}$ and Spec.

$$\textbf{Sob} \sim \textbf{SpatFrm}$$

L-valued topology

the category *L*-**Top**

With L a complete lattice and X a set, L^X is the complete lattice of all maps from X to L, called L-sets, in which

$$a \leq b \text{ in } L^X \quad \text{iff} \quad a(x) \leq b(x) \text{ for all } x \in X.$$

If $\alpha \in L$, the associated constant map is denoted $\underline{\alpha}$.

If $f : X \to Y$ and $b \in L^Y$ we let $f^{-1}(b) = b \circ f \in L^X$.

An L-valued topological space (shortly, an L-topological space) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the L-valued topology or L-topology on the set X) closed under finite meets and arbitrary joins.

Given two L-topological spaces $(X, \tau), (Y, \sigma)$ a map $f : X \to Y$ is an L-continuous map if the correspondence $f^{-1}(b)$ maps σ into τ. The resulting category will be denoted by L-**Top**.

"*It is a natural and interesting question whether or not it is possible to establish a category to play the same role with respect to a given notion of fuzzy topology as that locales play for topological spaces.*"

D. Zhang, Y. Liu, *L-fuzzy version of Stone's representation theorem for distributive lattices*, Fuzzy Sets and Systems 76 (1995) 259-270.

The ι_L functor

The iota functor ι_L was originally introduced by Lowen with $L = [0,1]$ and later on extended by Kubiak to an arbitrary complete lattice.

Let L be a complete lattice and X be a set. For a fixed $\alpha \in L \setminus \{1\}$ and let $a \in L^X$, we denote

$$[a \not\leq \alpha] = \{x \in X : a(x) \not\leq \alpha\}.$$

This defines a map $\iota_\alpha : L^X \to \mathbf{2}^X$ by $\iota_\alpha(a) = [a \not\leq \alpha]$.
Now, given an L-topology τ on X, we consider the topology

$$\iota_L(\tau) = \langle \{\iota_\alpha(\tau) : \alpha \in L\} \rangle = \langle \{\iota_\alpha(a) : a \in \tau, \alpha \in L\} \rangle.$$

This defines a functor ι_L : L-**Top** $\to$ **Top** by

$$\iota_L(X,\tau) = (X, \iota_L(\tau)), \qquad \iota_L(h) = h.$$

The ι_L functor

chain valued frames

Let L be a complete chain. Then $\iota_\alpha(a) = [a > \alpha]$.

We can consider the system of frame homomorphisms

$$\big(\iota_\alpha : \tau \to \iota_L(\tau) \mid \alpha \in L \setminus \{1\}\big).$$

The following are satisfied:

(F0) For each $\varnothing \neq S \subseteq L \setminus \{1\}$, we have that

$$\iota_{\bigwedge S}(a) = [a > \bigwedge S] = \bigcup_{\alpha \in S} [a > \alpha] = \bigcup_{\alpha \in S} \iota_\alpha(a).$$

(F1) $\iota_L(\tau) = \langle \bigcup_{\alpha \in L \setminus \{1\}} \iota_\alpha(\tau) \rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in τ, then $\iota_\alpha(a) \neq \iota_\alpha(b)$ for some $\alpha \in L \setminus \{1\}$. (collectionwise monomorphic)

The ι_L functor

chain valued frames

Let L be a complete chain, A a frame and $(X, \mathcal{T})$ a topological space and let

$$\big(\varphi_\alpha : A \to \mathcal{T} \mid \alpha \in L \setminus \{1\}\big)$$

be a system of frame homomorphisms satisfying (F0), (F1) and (F2).

Then there is a frame τ and an isomorphism $\kappa : A \to \tau$ such that:

1. (X, τ) is an L-space,
2. $\mathcal{T} = \iota_L(\tau)$,
3. $\iota_\alpha \circ \kappa = \varphi_\alpha$ for each $\alpha \in L \setminus \{1\}$.

The above discussion means that A is, up to frame isomorphism, an L-topology on X.

Chain valued frames

"*The notion of chain-valued frame, is introduced to be an abstraction of the distinctive properties of the system of level mappings from an L-topology* τ *into* $\iota_L(\tau)$*. These conditions, when L is a complete chain, were taken as axioms* (F0)*,* (F1) *and* (F2) *in order to define L-frames and the associated category L-*Frm*.*"

- A. Pultr, S.E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets,
  Kluwer Academic Publishers, 2003, pp. 153–187, (Chapter 6).
- A. Pultr, S.E. Rodabaugh, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations, Part II: Applications to lattice-valued topology*, Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558.

Chain valued frames

Let L be a complete chain. An L-frame A is a system

$$\left(\varphi^A_\alpha : A^{\mathfrak{u}} \to A^{\mathfrak{l}}\ \alpha \in L \setminus \{1\}\right)$$

of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{\alpha \in S} \varphi^A_\alpha$ for every $\varnothing \neq S \subseteq L \setminus \{1\}$.

(F1) $A^{\mathfrak{l}} = \langle \bigcup_{\alpha \in L\setminus\{1\}} \varphi^A_\alpha(A^{\mathfrak{u}}) \rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in $A^{\mathfrak{u}}$, then $\varphi^A_\alpha(a) \neq \varphi^A_\alpha(b)$ for some $\alpha \in L \setminus \{1\}$. (collectionwise monomorphic)

A. Pultr, S.E. Rodabaugh, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations, Part II: Applications to lattice-valued topology*, Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558.

Chain valued frames

An *L-frame morphism* $h : A \to B$ is an ordered pair of frame homomorphisms

$$h^{\mathfrak{u}} : A^{\mathfrak{u}} \to B^{\mathfrak{u}} \text{ and } h^{\mathfrak{l}} : A^{\mathfrak{l}} \to B^{\mathfrak{l}}$$

such that the following diagram is commutative for each $\alpha \in L \setminus \{1\}$

$$\begin{array}{ccc} A^{\mathfrak{u}} & \xrightarrow{\varphi_\alpha^A} & A^{\mathfrak{l}} \\ {\scriptstyle h^{\mathfrak{u}}}\downarrow & & \downarrow{\scriptstyle h^{\mathfrak{l}}} \\ B^{\mathfrak{u}} & \xrightarrow{\varphi_\alpha^B} & B^{\mathfrak{l}} \end{array}$$

The resulting category, with composition and identities component-wise in Frm, is denoted by *L*-Frm.

Chain valued frames

In the previous definitions L is assumed to be a complete chain, which seems to be quite a restrictive assumption.

"*During the preparation of the Volume, U. Höhle communicated to the authors of Chapter 6 that a complete chain is really only needed for its meet-irreducibles, and that for spatial L one also has meet-irreducibles which suffice for the constructions of Chapter 6.*"

U. Höhle and S.E. Rodabaugh, *Weakening the requirement that L be a complete chain*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, 2003, pp. 189–197, (Chapter 7).

Completely distributive lattices

Given $\alpha, \beta \in L$, we say that α is way below β, in symbols $\alpha \ll \beta$, if and only if

$$\left.\begin{array}{l} S \subseteq L \text{ and} \\ \beta \leq \bigvee S \end{array}\right\} \implies \text{there exist } \gamma_1, \ldots \gamma_n \in S \text{ such that } \alpha \leq \vee_{i=1}^{n} \gamma_i.$$

Recall that L is continuous if and only if the way-below relation is approximating, i.e., if and only if

$$\alpha = \bigvee \{\beta \in L : \beta \ll \alpha\} \quad \text{for each } \alpha \in L.$$

Let $\alpha, \beta, \gamma, \delta \in L$, then:

(1) $\alpha \ll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \ll \gamma \leq \delta$ implies $\alpha \ll \delta$.

(3) If L is continuous, then $\alpha \ll \beta$ implies $\alpha \ll \gamma \ll \beta$ for some $\gamma \in L$

Completely distributive lattices

We shall be particularly interested in the opposite relation of the way-below relation in the lattice L^{op}, denoted by $\lll$.

Namely, given $\alpha, \beta \in L$ we have $\alpha \lll \beta$ if

$$\left.\begin{array}{l} S \subseteq L \text{ and} \\ \bigwedge S \leq \alpha \end{array}\right\} \implies \text{ there exist } \gamma_1, \ldots \gamma_n \in S \text{ such that } \wedge_{i=1}^n \gamma_i \leq \beta.$$

For each $\alpha \in L$ we write $\twoheaduparrow\alpha = \{\beta \in L : \alpha \lll \beta\}$.

Then we have that L^{op} is alertcontinuous if and only if it satisfies

$$\alpha = \bigwedge\{\beta \in L : \alpha \lll \beta\} = \bigwedge \twoheaduparrow\alpha \quad \text{for all } \alpha \in L.$$

The following properties of the binary relation $\lll$ will be needed:

(1) $\alpha \lll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \lll \gamma \leq \delta$ implies $\alpha \lll \delta$.

Completely distributive lattices

A lattice is called completely distributive iff it is complete and for any familiy $\{x_{j,k} : j \in J, k \in K(j)\}$ in L the identity

$$\bigwedge_{j\in J} \bigvee_{k\in K(j)} x_{j,k} = \bigvee_{f\in M} \bigwedge_{j\in J} x_{j,f(j)} \tag{CD}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$.

We recall now the following result:

Let L be a complete lattice. Then the following are equivalent:

(1) L is completely distributive.

(2) L is a spatial frame and L^{op} is continuous.

Completely distributive lattices

Let L be a complete lattice. Then the following are equivalent:

(1) L is completely distributive.

(2) L satisfies the following two properties:

(i) L is a spatial frame, i.e.
$\alpha = \bigwedge (\uparrow\alpha \cap \mathrm{Spec}\, L)$ for each $\alpha \in L$,

(ii) L^{op} is continuous., i.e.
$\alpha = \bigwedge \twoheaduparrow\alpha$ for all $\alpha \in L$. $p = \bigwedge (\twoheaduparrow p \cap \mathrm{Spec}\, L)$ for each $p \in \mathrm{Spec}\, L$.

The ι_L functor — completely distributive lattices

Let L be a completely distributive lattice. The mapping $\iota_p : \tau \to \iota_L(\tau)$ is a frame morphism for each $p \in \mathrm{Spec}\, L$ (this is not true in general if p fails to be prime). Consider the system of frame morphisms

$$\big(\iota_p : \tau \to \iota_L(\tau) \mid p \in \mathrm{Spec}\, L\big).$$

(F0)' $\iota_p = \bigvee_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} \iota_q$ for each $p \in \mathrm{Spec}\, L$.

(F1) $\iota_L(\tau) = \Big\langle \bigcup_{p \in \mathrm{Spec}\, L} \iota_p(\tau) \Big\rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in τ then $\iota_p(a) \neq \iota_p(b)$ for some $p \in \mathrm{Spec}\, L$. (collectionwise monomorphic)

The ι_L functor — completely distributive lattices

Let L be a completely distributive lattice, A a frame and $(X, \mathcal{T})$ a topological space and let

$$\left(\varphi_p : A \to \mathcal{O}X \mid p \in \mathrm{Spec}\, L\right)$$

be a system of frame homomorphisms satisfying (F0)', (F1) and (F2).

Then there is a frame τ and an isomorphism $\kappa : A \to \tau$ such that:

1. (X, τ) is an L-space,
2. $\mathcal{T} = \iota_L(\tau)$,
3. $\iota_p \circ \kappa = \varphi_p$ for each $p \in \mathrm{Spec}\, L$.

The above discussion means that A is, up to frame isomorphism, an L-topology on X.

The ι_L functor — completely distributive lattices

A subset S of a complete lattice is downdirected if it is non-empty and for any $\alpha, \beta \in S$ there is some $\gamma \in S$ such that $\gamma \leq \alpha \wedge \beta$.

In a complete chain L any non-empty subset is downdirected and $\mathrm{Spec}\, L = L \setminus \{1\}$ and hence $\left(\varphi_p^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid p \in \mathrm{Spec}\, L\right)$ is a system of frame morphisms, then axiom (F0) can be equivalently stated as:

(F0) $\varphi^A_{\bigwedge S} = \bigvee\limits_{s \in S} \varphi^A_s$ for each downdirected $\varnothing \neq S \subseteq \mathrm{Spec}\, L$.

For a completely distributive lattice L the following are equivalent

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $\varnothing \neq S \subseteq \mathrm{Spec}\, L$.

(F0)' $\varphi^A_p = \bigvee_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} \varphi^A_q$ for each $p \in \mathrm{Spec}\, L$.

L-valued frames

Let L be a completely distributive lattice. An L-frame A is a system

$$\left(\varphi_p^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}}\ p \in \mathrm{Spec}\, L\right)$$

of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

(F0) $\varphi_p^A = \bigvee_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} \varphi_q^A$ for every $p \in \mathrm{Spec}\, L$.

(F1) $A^{\mathfrak{l}} = \langle \bigcup_{p \in \mathrm{Spec}\, L} \varphi_p^A(A^{\mathfrak{u}}) \rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in $A^{\mathfrak{u}}$ then $\varphi_p^A(a) \neq \varphi_p^A(b)$ for some $p \in \mathrm{Spec}\, L$. (collectionwise monomorphic)

J.G.G., U. Höhle, M.A. de Prada Vicente, *On lattice-valued frames*, Fuzzy Sets and Systems 159 (2010) 1022–1030.

L-valued frames

An L-frame morphism $h : A \to B$ is an ordered pair of frame homomorphisms

$$h^{\mathfrak{u}} : A^{\mathfrak{u}} \to B^{\mathfrak{u}} \text{ and } h^{\mathfrak{l}} : A^{\mathfrak{l}} \to B^{\mathfrak{l}}$$

such that the following diagram is commutative for each $p \in \operatorname{Spec} L$

$$\begin{array}{ccc} A^{\mathfrak{u}} & \xrightarrow{\varphi_p^A} & A^{\mathfrak{l}} \\ {\scriptstyle h^{\mathfrak{u}}}\downarrow & & \downarrow{\scriptstyle h^{\mathfrak{l}}} \\ B^{\mathfrak{u}} & \xrightarrow{\varphi_p^B} & B^{\mathfrak{l}} \end{array}$$

The resulting category, with composition and identities component-wise in Frm, is denoted by L-Frm.

L-valued frames

- The new notion coincides with that of Pultr and Rodabaugh when L is a complete chain.
- All the results proved by Pultr and Rodabaugh regarding the category L-Frm for a complete chain can now be extended to this new setting.

In particular if we denote by $\mathfrak{F}_0$ and $\mathfrak{F}_1$ the categories in which the objects are frame morphisms for which only (F0) (resp. (F0) and (F1)) is (resp. are) satisfied.Then

- $\mathfrak{F}_0$ is complete and cocomplete and each of the forgetful functors $\mathcal{U}^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{l}} : \mathfrak{F}_0 \to \mathrm{Frm}$ preserves all limits and colimits.
- $\mathfrak{F}_1$ is complete and cocomplete.
- L-Frm is complete and cocomplete.

L-valued frames possible extensions

Our work was motivated by the question stated in the papers of Pultr and Rodabaugh, when the authors suggest that relaxing the condition of a complete chain is a significant question.

We have already specified an answer by proving that the condition of a complete chain can be relaxed to a completely distributive lattice and that the completeness and cocompleteness of $\mathfrak{F}_0$, $\mathfrak{F}_1$ and L-Frm are still satisfied.

In this context the natural question arises whether weakening of complete distributivity is still possible. As an answer to this question we show that complete distributivity is necessary for the property that for every L-topological space (X, τ) the system

$$\big(\iota_p : \tau \to \iota_L(\tau) \mid p \in \mathrm{Spec}\, L\big)$$

of frame homomorphisms ι_p satisfies (F0) and (F2).

L-valued frames

possible extensions

Let L be a frame, (X, τ) an L-topological space and

$$\left(\iota_p : \tau \to \iota_L(\tau) \mid p \in \mathrm{Spec}\, L\right)$$

the system of frame morphisms determined by the ι-functor. Then:

- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ satisfies axiom (F0) for each (X, τ), then

$$p = \bigwedge \left(\twoheaduparrow p \cap \mathrm{Spec}\, L\right) \quad \text{for each } p \in \mathrm{Spec}\, L.$$

- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ satisfies axiom (F2) for each (X, τ), then L is spatial.
- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ is an L-frame for each (X, τ), then L is a completely distributive lattice.