Lattice-valued categorically-algebraic topology

Sergejs Solovjovs[1,2]

[1]Department of Mathematics, University of Latvia
e-mail: sergejs.solovjovs@lu.lv

[2]Institute of Mathematics and Computer Science, University of Latvia
e-mail: sergejs.solovjovs@lumii.lv

Alexander Shostak's Workshop

Department of Mathematics, University of Latvia
Riga, Latvia
November 30, 2010

Outline

Motivation and beyond

Challenging problem

There exist a lot of different approaches to lattice-valued topology. Every new theory pursues its own path and the intercommunication means are scarce. As a consequence, the results of every researcher are valid in his own setting only, causing reinvention of the proof of the standard topological properties in each new case.

Possible solution

Find an approach to lattice-valued topology, which will incorporate the majority of the current lattice-valued topological frameworks.

Natural tools

Due to its generality, category theory seems to be the perfect tool. To avoid undue abstractedness, universal algebra is advisable.

Motivating approach to topology

Modern topology can be based in three notions:

1. The powerset theory, which is the functor $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ from the category of sets to the dual of the variety of complete Boolean algebras: $(X \xrightarrow{f} Y)^{\leftarrow} = \mathbf{2}^X \xrightarrow{(f^{\leftarrow})^{op}} \mathbf{2}^Y$, $f^{\leftarrow}(\alpha) = \alpha \circ f$.
2. The topological theory, which is the obvious forgetful functor $\mathbf{CBAlg} \xrightarrow{\|-\|} \mathbf{Frm}$ to the category of frames.
3. The category **Top** of topological spaces and continuous maps, whose objects are pairs (X, τ), with τ (topology) a subframe of $\|\mathbf{2}^X\|$, and whose morphisms $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are maps $X \xrightarrow{f} Y$, with $\|f^{\leftarrow}\|(\alpha) \in \tau$ for every $\alpha \in \sigma$ (continuity).

Categorically-algebraic approach to topology

We propose the following generalization:

1. A categorically-algebraic (catalg) powerset theory is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual of a variety of algebras $\mathbf{A}$.
2. Given a catalg powerset theory $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ and a reduct $\mathbf{B}$ of $\mathbf{A}$, with the forgetful functor $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$, the induced catalg topological theory is the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op} = \mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$.
3. Given a topological theory T, $\mathbf{Top}(T)$ is the category, whose objects (catalg topological spaces) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ (catalg topology) is a subalgebra of $T(X)$, and whose morphisms $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $(Tf)^{op}(\alpha) \in \tau$ for every $\alpha \in \sigma$ (catalg continuity).

Categories of catalg theories and structures

Definition 1

- **TpThr** is the (quasi)category, the objects of which are catalg topological theories $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, and morphisms $T_1 \xrightarrow{\langle F,\Phi,\eta\rangle} T_2$ comprise a pair of functors $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$, $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ and a natural transformation $T_2 \circ F \xrightarrow{\eta} \Phi^{op} \circ T_1$.
- $\mathbf{TpThr}_s$ is the subcategory of **TpThr** with the same objects, and those morphisms $\langle F, \Phi, \eta\rangle$, where Φ preserves surjective homomorphisms.

Definition 2

TpSpc is the (quasi)category of the categories of the form $\mathbf{Top}(T)$ and functors between them.

From catalg theories to catalg structures

Let $\mathbf{Set} \xrightarrow{(-)^{\rightarrow}} \mathbf{Set}$ be the covariant powerset functor.

Theorem 3

There exists a functor $\mathbf{TpThr}_s \xrightarrow{\mathrm{Top}} \mathbf{TpSpc}$ *given by*

$$\mathrm{Top}(T_1 \xrightarrow{\langle F,\Phi,\eta\rangle} T_2) = \mathbf{Top}(T_1) \xrightarrow{\mathrm{Top}\langle F,\Phi,\eta\rangle} \mathbf{Top}(T_2),$$

$$\mathrm{Top}\langle F,\Phi,\eta\rangle((X,\tau) \xrightarrow{f} (Y,\sigma)) =$$

$$(FX,(\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi(\tau))) \xrightarrow{Ff} (FY,(\eta_Y^{op} \circ \Phi e_\sigma)^{\rightarrow}(\Phi(\sigma))),$$

where $\tau \xhookrightarrow{e_\tau} T_1(X)$ *and* $\sigma \xhookrightarrow{e_\sigma} T_1(Y)$ *are the inclusions.*

Examples and advantages

Examples

1. Many classical approaches to topology.
2. The majority of lattice-valued approaches to topology.

Advantages

1. A common framework for many topological theories and the means for intercommunication between them are provided.
2. The border between crisp and many-valued developments is ultimately erased.
3. The amount of building blocks of the theory is at minimum.

Properties of catalg topology

Topological property

Catalg continuity of a morphism can be checked on the elements of the appropriately defined (sub)base.

Resulting categorical property

The category $\mathbf{Top}(T)$ is topological over its ground category $\mathbf{X}$.

Main developments of catalg theory

1. Catalg topological spaces try to adopt standard concepts of topology.
2. Catalg topological systems subsume catalg spaces and their underlying algebraic structures.
3. Catalg dualities provide a general machinery for obtaining topological representations of algebraic structures.
4. Catalg powerset theory extends the standard set-theoretic image and preimage operators induced by a map.
5. Catalg attachment generalizes the set-theoretic membership relation "$\in$".
6. Lattice-valued catalg topology fuzzifies catalg topology.

Research motivation

- Topological systems were introduced by S. Vickers as a common framework for both topological spaces and their underlying algebraic structures – locales.
- There were several investigations on relationships between topological systems and lattice-valued topology.
- The main result was that using fuzzy topological spaces on one side, you need fuzzy topological systems on the other.

Possible solution

Having catalg topological spaces in hand, it is natural to provide catalg topological systems.

Topological systems of S. Vickers

Definition 4

A topological system is a triple (X, A, κ), where X is a set, A is a frame and $A \xrightarrow{\kappa} \|\mathbf{2}^X\|$ is a frame homomorphism. Given topological systems (X_1, A_1, κ_1), (X_2, A_2, κ_2), a continuous map between them is a pair (f, φ), where $X_1 \xrightarrow{f} X_2$ is a map and $A_2 \xrightarrow{\varphi} A_1$ is a frame homomorphism, making the following diagram commute:

$$\begin{array}{ccc} A_2 & \xrightarrow{\varphi} & A_1 \\ {\scriptstyle \kappa_2}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ \|\mathbf{2}^{X_2}\| & \xrightarrow[\|f^{\leftarrow}\|]{} & \|\mathbf{2}^{X_1}\|. \end{array}$$

TopSys is the category of topological systems and continuous maps.

Catalg topological systems

Definition 5

Given a topological theory $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, **TopSys**$(T)$ is the comma category $(T \downarrow 1_{\mathbf{B}^{op}})$, whose objects (resp. morphisms) are called catalg topological systems (resp. catalg continuous morphisms).

Examples

1. Topological systems of S. Vickers, state property systems of D. Aerts, Chu spaces of V. Pratt, contexts of R. Wille.
2. Interchange systems and lattice-valued topological systems of J. T. Denniston, A. Melton and S. E. Rodabaugh.

Properties of catalg topological systems

Main categorical property

Under certain conditions, the category **TopSys**(T) is (essentially) algebraic over its ground category $\mathbf{X} \times \mathbf{B}^{op}$.

Relation between systems and spaces

The category **Top**(T) is isomorphic to a full (regular mono)-coreflective subcategory of the category **TopSys**(T).

Research motivation

Missing topological framework

- There exists an important framework of (L, M)-fuzzy topology of T. Kubiak and A. Šostak, which can not be accommodated within catalg topology.
- The main sticking point is the fact that (L, M)-fuzzy topology relies on lattice-valued algebras.

Possible solution

Extend the obtained catalg theory to lattice-valued algebras.

Algebras and homomorphisms

Definition 6

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers.

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-ary primitive operations on A).
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi \circ \omega_\lambda^A = \omega_\lambda^B \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.

- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects, $\mathcal{E}$-quotients.
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

Lattice-valued algebras and homomorphisms

Definition 7

Let **A** and **B** be varieties, with **B** having the variety **CSLat**($\bigvee$) of $\bigvee$-semilattices as a reduct, and let **L** be a subcategory of **B**.

- An **L-A**-algebra is a triple (A, μ, L), comprising an **A**-algebra A, an **L**-object L and a map $A \xrightarrow{\mu} L$ such that for every $\lambda \in \Lambda$ and every $\langle a_i \rangle_{n_\lambda} \in A^{n_\lambda}$, $\bigwedge_{i \in n_\lambda} \mu(a_i) \leq \mu(\omega_\lambda^A(\langle a_i \rangle_{n_\lambda}))$.
- An **L-A**-homomorphism $(A_1, \mu_1, L_1) \xrightarrow{(\varphi,\psi)} (A_2, \mu_2, L_2)$ is an **A** $\times$ **L**-morphism $(A_1, L_1) \xrightarrow{(\varphi,\psi)} (A_2, L_2)$ with $\psi \circ \mu_1 \leqslant \mu_2 \circ \varphi$.
- **L-A** is the category of **L-A**-algebras and **L-A**-homomorphisms.

Lattice-valued catalg spaces

Definition 8

Let $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ be a catalg topological theory, let $\mathbf{C}$ be an extension of $\mathbf{CSLat}(\bigvee)$, let $\mathbf{L}$ be a subcategory of $\mathbf{C}$. A lattice-valued (latval) catalg topological theory induced by T and $\mathbf{L}$ is the pair $(T, \mathbf{L})$.

Definition 9

Let $(T, \mathbf{L})$ be a latval catalg topological theory. $\mathbf{LTop}(T)$ is the category, whose objects (latval catalg spaces) are triples $(X, \mathcal{T}, L)$, with (X, L) in $\mathbf{X} \times \mathbf{L}^{op}$ and $(T(X), \mathcal{T}, L)$ an $\mathbf{L}$-$\mathbf{B}$-algebra (latval catalg topology), and whose morphisms $(X, \mathcal{T}, L) \xrightarrow{(f,\psi)} (Y, \mathcal{S}, M)$ are $\mathbf{X} \times \mathbf{L}^{op}$-morphisms $(X, L) \xrightarrow{(f,\psi)} (Y, M)$ with the requirement that $(T(X), \mathcal{T}, L) \xrightarrow{(Tf,\psi)} (T(Y), \mathcal{S}, M)$ gives an $(\mathbf{L}\text{-}\mathbf{B})^{op}$-morphism (latval catalg continuity).

Examples and advantages

Examples

1. Catalg topology.
2. (L, M)-fuzzy topological spaces of T. Kubiak and A. Šostak.
3. Variable-basis extension of (L, M)-fuzzy topological spaces of J. T. Denniston, A. Melton and S. E. Rodabaugh.
4. Fuzzy topology of C. Guido.

Main advantage

A fuzzification of the theory of catalg topology is provided.

Lattice-valued catalg systems

Definition 10

Let $(T, \mathbf{L})$ be a latval catalg topological theory. $\mathbf{LTopSys}(T)$ is the category, whose objects (latval catalg topological systems) are triples $(X, (B, \mu, L), \kappa)$ such that X is an $\mathbf{X}$-object and (B, μ, L) is an $(\mathbf{L}\text{-}\mathbf{B})^{op}$-object, whereas $T(X) \xrightarrow{\kappa} B$ is a $\mathbf{B}^{op}$-morphism (latval catalg satisfaction relation), and whose morphisms

$$(X, (B_1, \mu_1, L_1), \kappa_1) \xrightarrow{(f,(\varphi,\psi))} (X_2, (B_2, \mu_2, L_2), \kappa_2)$$

are $\mathbf{X} \times (\mathbf{L}\text{-}\mathbf{B})^{op}$-morphisms

$$(X_1, (B_1, \mu_1, L_1)) \xrightarrow{(f,(\varphi,\psi))} (X_2, (B_2, \mu_2, L_2))$$

such that $(X_1, B_1, \kappa_1) \xrightarrow{(f,\varphi)} (X_2, B_2, \kappa_2)$ is a $\mathbf{TopSys}(T)$-morphism (latval catalg continuity).

Properties of lattice-valued approach

Main categorical property of spaces

The category **LTop**(T) is topological over its ground category $\mathbf{X} \times \mathbf{L}^{op}$.

Relation between systems and spaces

If the underlying lattices of **L** are completely distributive, then the category **LTop**(T) is isomorphic to a full coreflective subcategory of the category **LTopSys**(T).

Conclusion

The main achievements

1. The presented framework incorporates the majority of modern approaches to lattice-valued topology.
2. The currently dominating in the fuzzy community approach of S. E. Rodabaugh appears to be "crisp" (goes in line with the crisp categorically-algebraic machinery), whereas the theory of T. Kubiak and A. Šostak results in a genuinely fuzzy approach (requires lattice-valued catalg topology).

Further proceedings

Develop the whole theory over (appropriately defined) varieties of lattice-valued algebras.

References I

- J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications, 2009.
- J. T. Denniston, A. Melton, and S. E. Rodabaugh, *Lattice-valued topological systems*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009, pp. 24–31.
- C. Guido, *Powerset Operators Based Approach to Fuzzy Topologies on Fuzzy Sets*, Topological and Algebraic Structures in Fuzzy Sets (S. E. Rodabaugh and E. P. Klement, eds.), Kluwer Academic Publishers, 2003, pp. 401–413.
- T. Kubiak and A. Šostak, *Foundations of the theory of* (L, M)*-fuzzy topological spaces*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009, pp. 70–73.

References II

- E. G. Manes, *Algebraic Theories*, Springer-Verlag, 1976.
- S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, Int. J. Math. Math. Sci. **2007** (2007), 1–71.
- A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
- S. Solovjovs, *Categorically-algebraic topology*, Abstracts of the International Conference on Algebras and Lattices, Charles University, Prague, 2010, pp. 20–22.
- S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.

Thank you for your attention!