Topologies modulo compatible ideals: set-theoretical study and representation in locale theory

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Outline of Study

Theoretical foundation

- Frame embedding modulo compatible ideal theorem
- Technique of finite induction for small sets

Topological study

- Generalized topological spaces (gt-spaces)
- Interior, continuity, cardinal invariants

Locale theory study

- Generalized spacial locale (gs-locales)
- Density, morphisms, classification

Representation theorems

Relations between categories and important subcategories

- Isomorphism of categories of gt-spaces and gs-locales
- Isomorphism of categories of $T_0$ topological spaces and crisp gs-locales
Let $X$ be a nonempty set, $I \subseteq 2^X$ be an ideal and $A, B$ be subsets of $X$. Then:

$$A \preceq B \iff A \setminus B \in I$$
and

$$A \approx B \iff (A \setminus B) \cup (B \setminus A) \in I.$$ 

Consider a family $\{A_k\}_{k \in K}$ of subsets of $X$ such that every $A_k$ is a subset modulo ideal of $B$. Is the union $A = \bigcup_{k \in K} A_k$ also a subset modulo ideal of $B$?
Theorem 12. Given a frame \((T, \lor, \land)\) and a complete Boolean lattice \((F, \cup, \cap, c)\) such that \(T \subseteq F\) and \(\text{id}: T \to F\) is an order embedding preserving zero. Then \(I = c(G)\) is the least ideal satisfying the following:

(i) for every \(U \subseteq T\) there is \(a \in I\) such that \(\lor U = (\bigcup U) \cup a\);

(ii) for every \(v, w \in T\) there is \(b \in I\) such that \(v \land w = (v \cap w) \setminus b\);

(iii) \(I \cap T = \{0\}\);

(iv) for every \(u, v \in T\), it holds that \(u \preceq v\) iff \(u \leq v\);

(v) \(I \sim T\).
Corollary 14. Let $X$ be a nonempty set. Assume that $T \subseteq 2^X$ forms a frame with respect to $\subseteq$ and $\emptyset, X \in T$. Then there exists the least ideal $I \subseteq 2^X$ such that:

(i) for every $\mathcal{U} \subseteq T$ holds $\bigvee \mathcal{U} \setminus \bigcup \mathcal{U} \in I$;

(ii) for every $V, W \in T$ holds $(V \cap W) \setminus (V \wedge W) \in I$;

(iii) $T \cap I = \{\emptyset\}$;

(iv) $U \preceq V$ implies $U \subseteq V$ for every $U, V \in T$;

(v) $U \approx V$ implies $U = V$ for every $U, V \in T$;

(vi) $I \sim T$.

Definition 15. Given a nonempty set $X$, a family $T \subseteq 2^X$ is called a generalized topology and the pair $(X, T)$ is called a generalized topological space provided that:

(GT1) $\emptyset, X \in T$;

(GT2) $(T, \subseteq)$ is a frame.
**Definition 25.** Given a gt-space \((X, T)\). The operators \(\ast \colon 2^X \to 2^X\) and \(\psi : 2^X \to 2^X\) are defined as follows, for all \(A \subseteq X\):

\[
\psi(A) = \{ x \in X \mid \text{exists } U \in T(x) \text{ such that } U \subseteq A \},
\]

\[
A^* = \{ x \in X \mid \text{for all } U \in T(x) \text{ it holds that } A \cap U \notin I \}.
\]

The operator \(\psi\) is called the interior operator and \(\ast\) is called the closure operator.

**Theorem 26.** In a gt-space \((X, T)\), the following hold for every \(A, B \subseteq X\):

(i) \(\psi(A) = \bigvee\{ U \subseteq X \mid U \subseteq A, U \text{ is open}\}\), and

\[
A^* = \bigwedge\{ C \subseteq X \mid A \subseteq C, C \text{ is closed}\};
\]

(ii) \(\psi(A) = X \setminus (X \setminus A)^*\);

(iii) \(A\) is open iff \(A = \psi(A)\), and \(A\) is closed iff \(A = A^*\);

(iv) \(\psi(X) = X\) and \(\emptyset^* = \emptyset\);

(v) \(\psi(A) \leq A \leq A^*\);

(vi) \(\psi(\psi(A)) = \psi(A)\) and \((B^*)^* = B^*\);

(vii) \(\psi(A \cap B) = \psi(A) \land \psi(B)\) and \((A \cup B)^* = A^* \lor B^*\).
Definition 27. Given gt-spaces \((X, T_X)\) and \((Y, T_Y)\). A mapping \(f: X \to Y\) is called a generalized continuous mapping provided that there exists a frame homomorphism \(h: T_Y \to T_X\) such that \(h(U) \approx f^{-1}(U)\) holds for every \(U \in T_Y\).
Theorem 28. Given gt-spaces \((X, T_X)\) and \((Y, T_Y)\), and a g-continuous mapping \(f: X \rightarrow Y\). Then the following hold:

(i) the corresponding frame homomorphism \(h: T_Y \rightarrow T_X\) is unique;

(ii) \(f^{-1}(B) \in I_X\) holds for all \(B \in I_Y\).

Proposition 31. Given gt-spaces \((X, T_X)\) and \((Y, T_Y)\), a base \(\mathcal{B} \subseteq T_Y\), and a mapping \(f: X \rightarrow Y\). Assume that

(1) \(f^{-1}(B) \in I_X\) holds for all \(B \in I_Y\),

(2) for every \(V \in \mathcal{B}\) there is \(V' \in T_X\) such that \(V' \approx f^{-1}(V) \subseteq V'\).

Then \(f\) is a g-continuous mapping.
Definition 38. Let \((X, T)\) be a gt-space. The operator \(N : T \to 2^X\) is called the normalization operator provided that, for every \(U \in T\):

\[
U^N = \{x \in U \mid U \cap V \neq \emptyset \text{ for all } V \in T(x)\}.
\]

The family \(T^N = \{U^N \mid U \in T\}\) is called the normalization of \(T\).

Proposition 39. Given a gt-space \((X, T)\), then the following hold:

(i) \(T^N\) is a frame, isomorphic to \(T\);

(ii) \((X, T^N)\) is a gt-space.

And the following conditions are equivalent:

(iii) \(T = T^N\);

(iv) \(U \cap V = \emptyset\) iff \(U \cap V = \emptyset\) for all \(U, V \in T\).
Network weight and weight

**Theorem 43.** In a gt-space \((X, T)\) it holds that \(nw(T) \leq w(T)\).

Lindelöf number and network weight

**Theorem 46.** In a gt-space \((X, T)\) it holds that \(l(T) \leq nw(T)\).

Suslin’s number, density and network weight

**Theorem 55.** In a gt-space \((X, T)\) it holds that \(c(T) \leq d(T) \leq nw(T)\).
Locale Theory

1. Given a frame $T$

2. Take the set $X$ as the family of all principal prime ideals of $T$

$$X = \text{pt}(T)$$

3. Construct the frame homomorphism $f: T \to 2^X$

$$f(u) = \{P \in \text{pt}(T) | u \notin P\}$$

4. Then the pair $(X, f(T))$ forms a sober topological space

$$(X, f(T))$$

$X = \text{pt}(T)$$
**Definition 58.** Given a frame $T$, write $L(T)$ for the family of proper lower subsets of $T$. We say that a subfamily $L \subseteq L(T)$ strongly separates the elements of $T$ iff for every $u, v \in T$ with $v \not\preceq u$ there exists $A \in L$ such that $u \in A$ and $v \not\in A$.

**Proposition 59.** Given a frame $T$ and a subfamily $L \subseteq L(T)$ that strongly separates the elements of $T$, define a mapping $p : T \to 2^L$ as follows, for all $u \in T$:

$$p(u) = \{ A \in L \mid u \not\in A \}.$$ 

Then the pair $(L, p(T))$ forms a $T_0$ gt-space and the mapping $p : T \to p(T)$ is a frame isomorphism.

**Definition 60.** Given a frame $T$ and a subfamily $L \subseteq L(T)$, we call the pair $(T, L)$ a generalized spatial locale provided that $L$ strongly separates the elements of $T$. 
Definition 63. Given a frame $T$, then the least cardinal number of the form $|S|$ where $S \subseteq L(T)$ strongly separates the elements of $T$ is called \textit{density} of $T$ and is denoted by $d(T)$.

Observations. The following hold:

1. $d(T) \leq |\text{Pr}(T)|$ holds for all frames $T$ (Example 61),
2. $d(T) = |\text{Pr}(T)|$ holds for finite frames $T$ (Proposition 62),
3. there is a frame $T$ such that $d(T) < |\text{Pr}(T)|$ (Example 64).
**Definition 65.** Given gs-locales $\langle T_Y, Y \rangle$ and $\langle T_X, X \rangle$. A pair of mappings $(h, f)$ is called a *homomorphism* provided that $h: T_Y \to T_X$ is a frame homomorphism, and $f: X \to Y$ is such that $p_X(h(U)) \Delta f^{-1}(p_Y(U)) \in I_X$ for every $U \in T_Y$.

\[
\begin{array}{ccc}
(T_Y, Y) & \xrightarrow{h} & (T_X, X) \\
p_Y & & \downarrow p_X \\
(Y, p(T_Y), I_Y) & \xleftarrow{f} & (X, p(T_X), I_X)
\end{array}
\]

The homomorphism $(h, f)$ is called an *isomorphism* provided that $f$ is a bijection and $h$ is a frame isomorphism.
**Theorem 69.** A $T_0$ gt-space $(X, T)$ is characterized up to g-homeomorphism by the gs-locale $(T, s(X))$.

$$
\begin{array}{ccc}
\text{gt-space} & (X, T) & \text{g-homeomorphism} \\
& \leftarrow & (s(X), p(T)) \\
& \downarrow s & \downarrow p \\
gs-locale & (T, s(X)) & \\
\end{array}
$$

**Theorem 70.** A gs-locale $(T, L)$ is characterized up to isomorphism by the gt-space $(L, p(T))$.

$$
\begin{array}{ccc}
\text{gs-locale} & (T, L) & \text{isomorphism} \\
& \leftarrow & (p(T), s(L)) \\
& \downarrow p & \downarrow s \\
gt-space & (L, p(T)) & \\
\end{array}
$$
Representation of gt-spaces and gslocales

Spatial locales $(T, pt(T))$ ↔ Sober spaces

gs-locales $(T, L)$ ↔ $T_0$ gt-spaces

Crisp gs-locales $(T, L \subseteq pt(T))$ ↔ $T_0$ spaces
Main results

- Frame embedding modulo compatible ideal theorem
- Notion of generalized topological spaces
  - Notions of interior and closure operators
  - Notion of generalized continuous mapping and related theorems
- Notion of generalized spacial locale
- Isomorphism of categories of gt-spaces and gs-locales
  - Isomorphism of categories of $T_0$ topological spaces and crisp gs-locales
Thank you for your attention!