Topologies modulo compatible ideals: set-theoretical study and representation in locale theory

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Outline of Study

Theoretical foundation

★ Frame embedding modulo compatible ideal theorem
★ Technique of finite induction for small sets

Topological study

★ Generalized topological spaces (gt-spaces)
★ Interior, continuity, cardinal invariants

Locale theory study

★ Generalized spacial locale (gs-locales)
★ Density, morphisms, classification

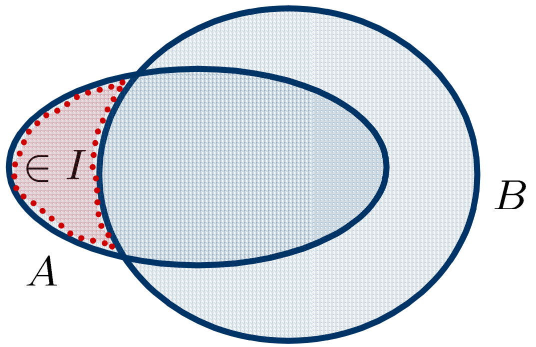
Representation theorems
Relations between categories and important subcategories

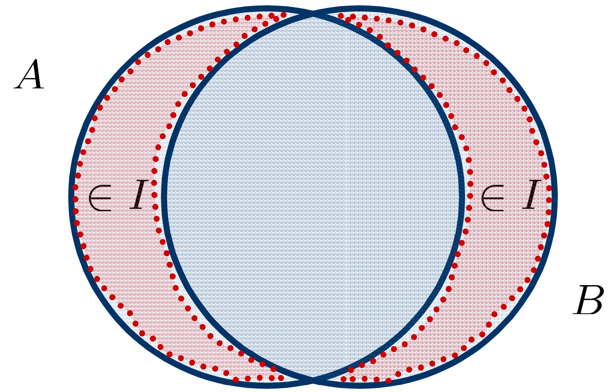
★ Isomorphism of categories of gt-spaces and gs-locales
★ Isomorphism of categories of T_0 topological spaces and crisp gs-locales

Relations Generated by Ideal. Compatibility

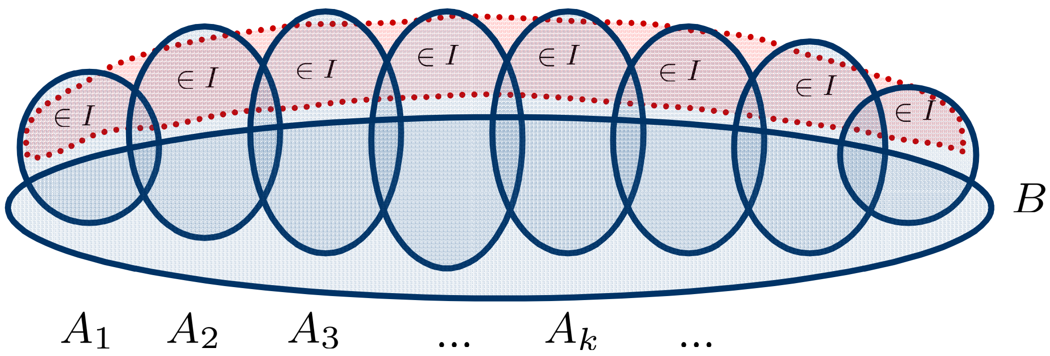
Let X be a nonempty set, $I \subseteq 2^X$ be an ideal and A, B be subsets of X. Then:

$$\boldsymbol{A} \preceq \boldsymbol{B} \text{ iff } A \setminus B \in I \qquad \text{and} \qquad \boldsymbol{A} \approx \boldsymbol{B} \text{ iff } (A \setminus B) \cup (B \setminus A) \in I.$$





Consider a family $\{A_k\}_{k \in K}$ of subsets of X such that every A_k is a subset modulo ideal of B. **Is the union $A = \bigcup_{k \in K} A_k$ also a subset modulo ideal of B?**



Theorem 12. Given a frame $(T, \vee, \wedge)$ and a complete Boolean lattice $(F, \cup, \cap, {}^c)$ such that $T \subseteq F$ and $\mathrm{id}\colon T \to F$ is an order embedding preserving zero. Then $I = c(G)$ is the least ideal satisfying the following:

(i) for every $U \subseteq T$ there is $a \in I$ such that $\bigvee U = (\bigcup U) \cup a$;

(ii) for every $v, w \in T$ there is $b \in I$ such that $v \wedge w = (v \cap w) \setminus b$;

(iii) $I \cap T = \{0\}$;

(iv) for every $u, v \in T$, it holds that $u \preceq v$ iff $u \leq v$;

(v) $I \sim T$.

Corollary 14. Let X be a nonempty set. Assume that $T \subseteq 2^X$ forms a frame with respect to $\subseteq$ and $\varnothing, X \in T$. Then there exists the least ideal $I \subseteq 2^X$ such that:

(i) for every $\mathcal{U} \subseteq T$ holds $\bigvee \mathcal{U} \setminus \bigcup \mathcal{U} \in I$;

(ii) for every $V, W \in T$ holds $(V \cap W) \setminus (V \wedge W) \in I$;

(iii) $T \cap I = \{\varnothing\}$;

(iv) $U \preceq V$ implies $U \subseteq V$ for every $U, V \in T$;

(v) $U \approx V$ implies $U = V$ for every $U, V \in T$;

(vi) $I \sim T$.

Definition 15. Given a nonempty set X, a family $T \subseteq 2^X$ is called a *generalized topology* and the pair (X, T) is called a *generalized topological space* provided that:

(GT1) $\varnothing, X \in T$;

(GT2) $(T, \subseteq)$ is a frame.

Definition 25. Given a gt-space (X, T). The operators $^*\colon 2^X \to 2^X$ and $\psi\colon 2^X \to 2^X$ are defined as follows, for all $A \subseteq X$:

$$\psi(A) = \{x \in X \mid \text{ exists } U \in T(x) \text{ such that } U \preceq A\},$$
$$A^* = \{x \in X \mid \text{ for all } U \in T(x) \text{ it holds that } A \cap U \notin I\}.$$

The operator ψ is called *the interior operator* and * is called *the closure operator.*

Theorem 26. In a gt-space (X, T), the following hold for every $A, B \subseteq X$:

(i) $\psi(A) = \bigvee\{U \subseteq X \mid U \preceq A,\ U \text{ is open}\}$, and

$A^* = \bigwedge\{C \subseteq X \mid A \preceq C,\ C \text{ is closed}\}$;

(ii) $\psi(A) = X \setminus (X \setminus A)^*$;

(iii) A is open iff $A = \psi(A)$, and A is closed iff $A = A^*$;

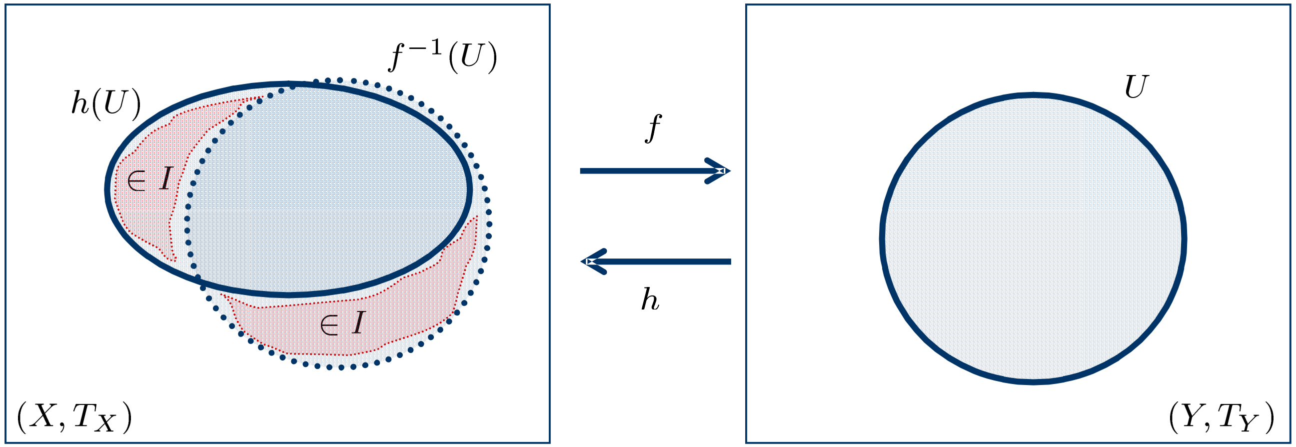
(iv) $\psi(X) = X$ and $\varnothing^* = \varnothing$;

(v) $\psi(A) \preceq A \preceq A^*$;

(vi) $\psi(\psi(A)) = \psi(A)$ and $(B^*)^* = B^*$;

(vii) $\psi(A \cap B) = \psi(A) \wedge \psi(B)$ and $(A \cup B)^* = A^* \vee B^*$.

Definition 27. Given gt-spaces (X, T_X) and (Y, T_Y). A mapping $f\colon X \to Y$ is called a *generalized continuous mapping* provided that there exists a frame homomorphism $h\colon T_Y \to T_X$ such that $h(U) \approx f^{-1}(U)$ holds for every $U \in T_Y$.



Theorem 28. Given gt-spaces (X, T_X) and (Y, T_Y), and a g-continuous mapping $f\colon X \to Y$. Then the following hold:

(i) the corresponding frame homomorphism $h\colon T_Y \to T_X$ is unique;

(ii) $f^{-1}(B) \in I_X$ holds for all $B \in I_Y$.

Proposition 31. Given gt-spaces (X, T_X) and (Y, T_Y), a base $\mathcal{B} \subseteq T_Y$, and a mapping $f\colon X \to Y$. Assume that

(1) $f^{-1}(B) \in I_X$ holds for all $B \in I_Y$,

(2) for every $V \in \mathcal{B}$ there is $V' \in T_X$ such that $V' \approx f^{-1}(V) \subseteq V'$.

Then f is a g-continuous mapping.

Definition 38. Let (X, T) be a gt-space. The operator $^N : T \to 2^X$ is called the *normalization operator* provided that, for every $U \in T$:

$$U^N = \{x \in U \mid U \wedge V \neq \varnothing \text{ for all } V \in T(x)\}.$$

The family $T^N = \{U^N \mid U \in T\}$ is called the *normalization of* T.

Proposition 39. Given a gt-space (X, T), then the following hold:

(i) T^N is a frame, isomorphic to T;

(ii) (X, T^N) is a gt-space.

And the following conditions are equivalent:

(iii) $T = T^N$;

(iv) $U \wedge V = \varnothing$ iff $U \cap V = \varnothing$ for all $U, V \in T$.

Network weight and weight

Theorem 43. In a gt-space (X, T) it holds that $nw(T) \leq w(T)$.

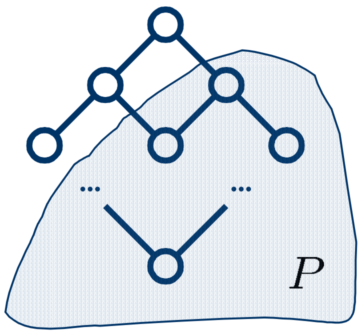
Lindelöf number and network weight

Theorem 46. In a gt-space (X, T) it holds that $l(T) \leq nw(T)$.

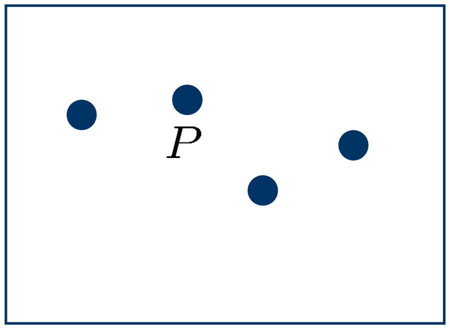
Suslin's number, density and network weight

Theorem 55. In a gt-space (X, T) it holds that $c(T) \leq d(T) \leq nw(T)$.

1 Given a frame T



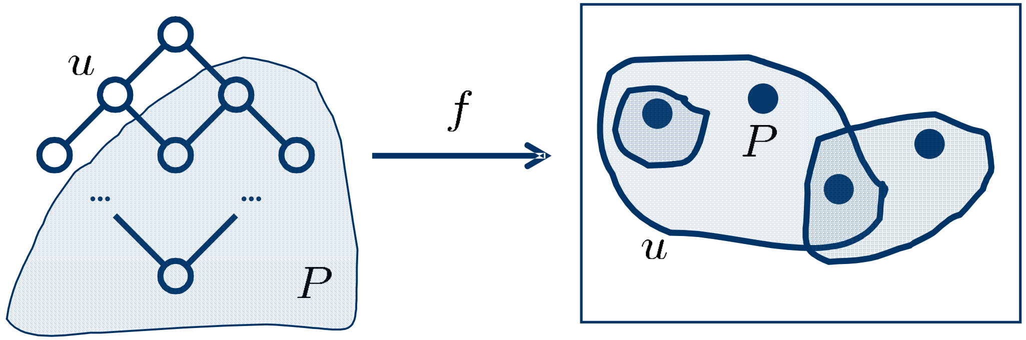
2 Take the set X as the family of all principal prime ideals of T



$X = \mathbf{pt}(T)$

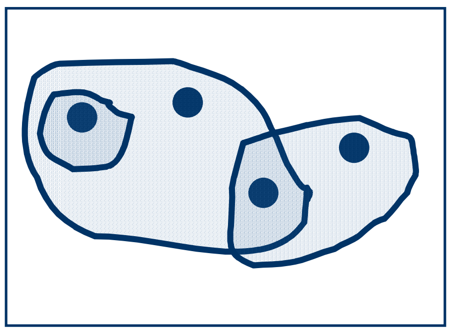
3 Construct the frame homomorphism $f : T \to 2^X$

$f(u) = \{P \in \mathbf{pt}(T) \mid u \notin P\}$



$X = \mathbf{pt}(T)$

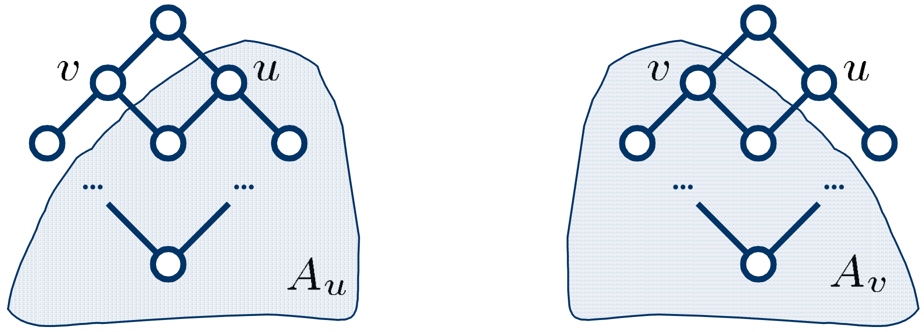
4 Then the pair $(X, f(T))$ forms a sober topological space



$(X, f(T))$

Notion of generalized spacial locale (gs-locale)

Definition 58. Given a frame T, write $\mathbf{L}(T)$ for the family of proper lower subsets of T. We say that a subfamily $L \subseteq \mathbf{L}(T)$ *strongly separates* the elements of T iff for every $u, v \in T$ with $v \nleq u$ there exists $A \in L$ such that $u \in A$ and $v \notin A$.

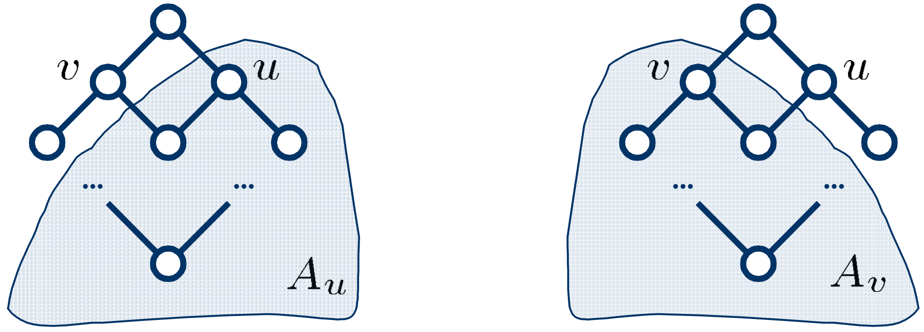


Proposition 59. Given a frame T and a subfamily $L \subseteq \mathbf{L}(T)$ that strongly separates the elements of T, define a mapping $p\colon T \to 2^L$ as follows, for all $u \in T$:

$$p(u) = \{A \in L \mid u \notin A\}.$$

Then the pair $(L, p(T))$ forms a T_0 gt-space and the mapping $p\colon T \to p(T)$ is a frame isomorphism.

Definition 60. Given a frame T and a subfamily $L \subseteq \mathbf{L}(T)$, we call the pair (T, L) a *generalized spatial locale* provided that L strongly separates the elements of T.



Definition 63. Given a frame T, then the least cardinal number of the form $|S|$ where $S \subseteq \mathbf{L}(T)$ strongly separates the elements of T is called *density of* T and is denoted by $d(T)$.

Observations. The following hold:

1. $d(T) \leq |\mathbf{Pr}(T)|$ holds for all frames T (Example 61),
2. $d(T) = |\mathbf{Pr}(T)|$ holds for finite frames T (Proposition 62),
3. there is a frame T such that $d(T) < |\mathbf{Pr}(T)|$ (Example 64).

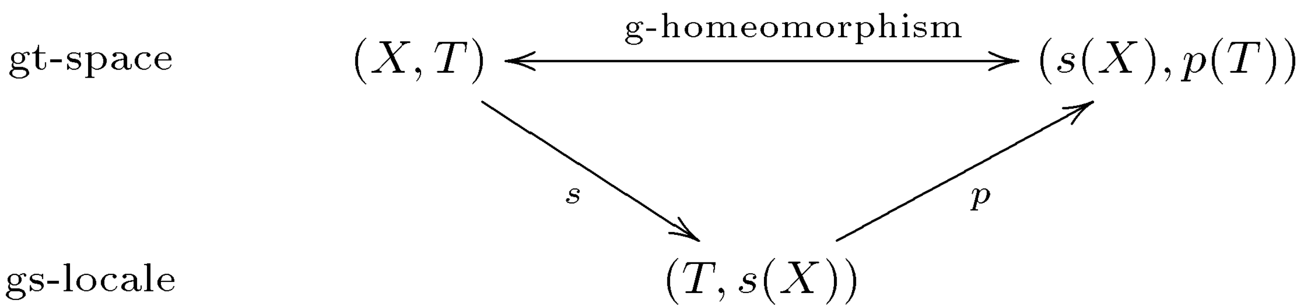
Homomorphisms of gs-locales

Definition 65. Given gs-locales (T_Y, Y) and (T_X, X). A pair of mappings (h, f) is called a *homomorphism* provided that $h \colon T_Y \to T_X$ is a frame homomorphism, and $f \colon X \to Y$ is such that $p_X(h(U)) \Delta f^{-1}(p_Y(U)) \in I_X$ for every $U \in T_Y$.

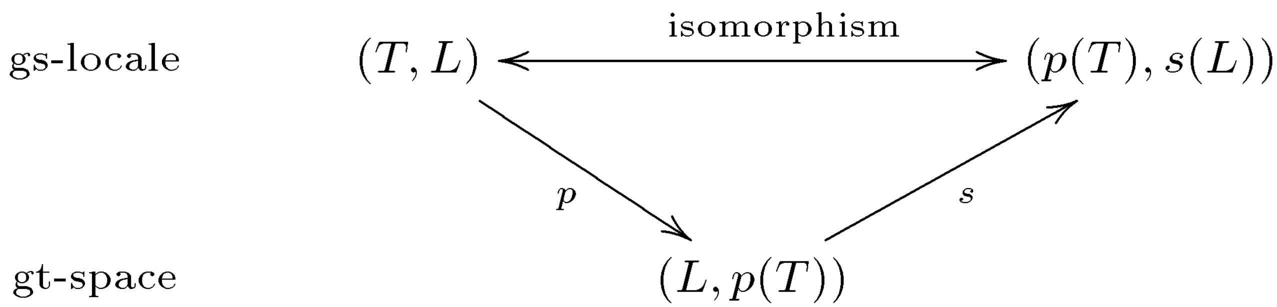
$$\begin{array}{ccc} (T_Y, Y) & \xrightarrow{\quad h \quad} & (T_X, X) \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_X} \\ (Y, p(T_Y), I_Y) & \xleftarrow[\quad f \quad]{} & (X, p(T_X), I_X) \end{array}$$

The homomorphism (h, f) is called an *isomorphism* provided that f is a bijection and h is a frame isomorphism.

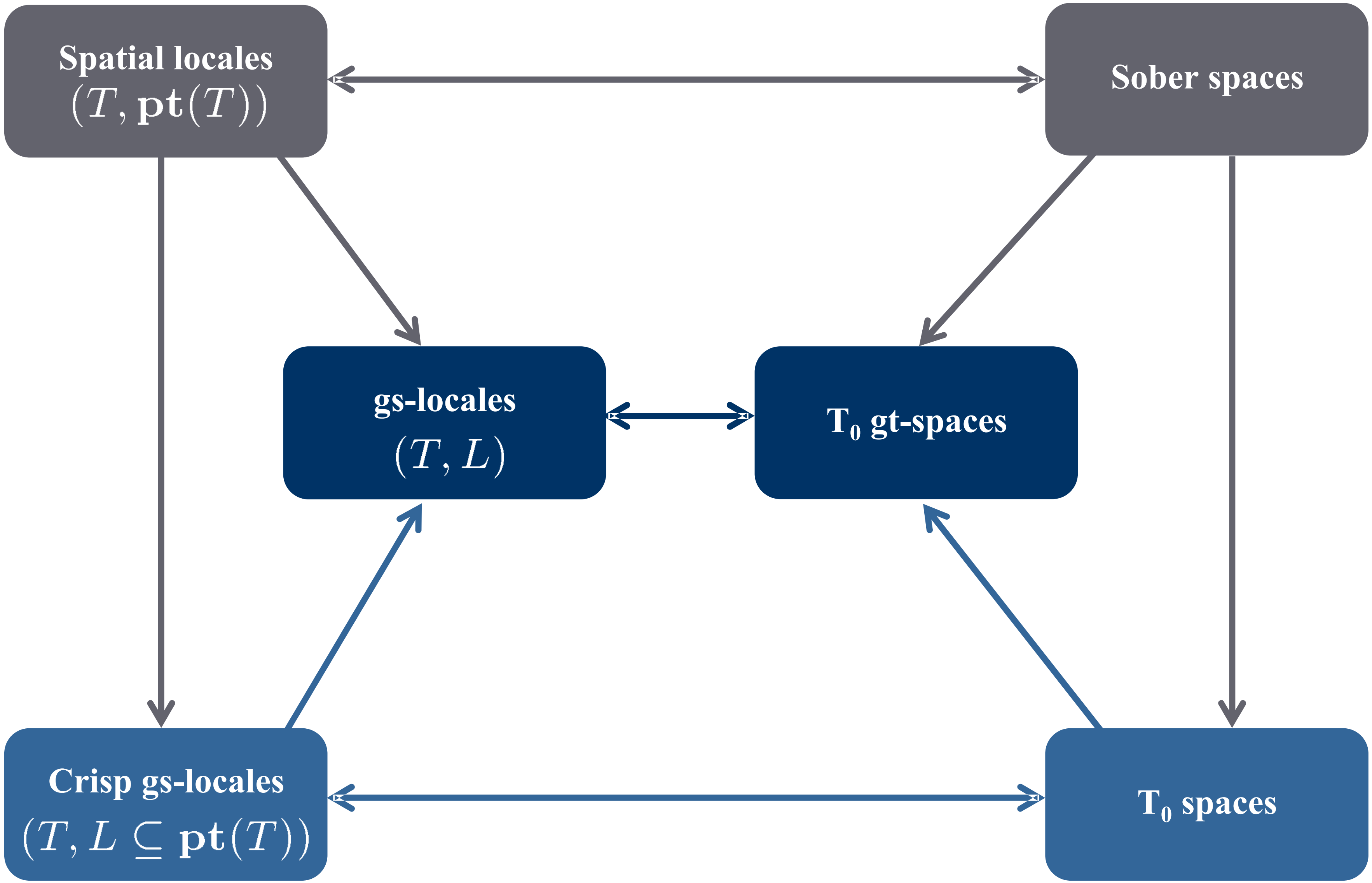
Theorem 69. A T_0 gt-space (X,T) is characterized up to g-homeomorphism by the gs-locale $(T,s(X))$.



Theorem 70. A gs-locale (T,L) is characterized up to isomorphism by the gt-space $(L,p(T))$.



Represenation of gt-spaces and gs-locales



Main results

★ Frame embedding modulo compatible ideal theorem

★ Notion of generalized topological spaces
★ Notions of interior and closure operators
★ Notion of generalized continuous mapping and related theorems

★ Notion of generalized spacial locale

★ Isomorphism of categories of gt-spaces and gs-locales
★ Isomorphism of categories of T_0 topological spaces and crisp gs-locales

Thank you for your attention!