Topological categories versus categorically-algebraic topology

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Alexander Shostak's Workshop

Department of Mathematics, University of Latvia
Riga, Latvia

December 14, 2011

Outline

Categorically-algebraic topology

- Lattice-valued topology is an approach to topology, which is based in lattice-valued sets of L. A. Zadeh and J. A. Goguen.
- There exist many different lattice-valued topological frameworks, e.g., categorical topological theories of S. E. Rodabaugh.
- Categorically-algebraic (catalg) topology is an approach to topology, which is based in category theory and universal algebra.
- Catalg topology provides a common setting for the majority of lattice-valued topological frameworks and gives convenient means of interaction between different topological theories.

Universal topology

- Categorical topology has been initiated by H. Herrlich in 1971.
- Based in category theory, it is mostly concerned with the study of topological categories and their relationships to each other.
- In 1983, H. Herrlich started its branch called universal topology, to study topological categories via a 2-step approach: constructing fundamental topological categories first and then, singling out topological subcategories by topological (co-)axioms.

Main result

A concrete category is fibre-small and topological iff it is definable by a class of topological co-axioms in a functor-costructured category.

Topological theories of O. Wyler

- In 1971, O. Wyler introduced the concept of topological theory.
- Based in category theory, the notion used the methods and results of categorical algebra in general topology.

Main result

Every fibre-small topological category is concretely isomorphic to the category of models of some topological theory.

Catalg topology versus categorical topology

- There has been an attempt to compare topological theories of S. E. Rodabaugh and O. Wyler, which claimed to resolve completely the relationships between them.
- Since the claimed resolution is neither complete nor error-free, this talk gives a detailed account on the relationships between catalg topology and categorical topology.

Main result

A concrete category is fibre-small and topological iff it is isomorphic to a full subcategory of some category of catalg topological structures, which is definable by topological co-axioms in it.

Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a class of cardinal numbers.

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-ary primitive operations on A).
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi \circ \omega_\lambda^A = \omega_\lambda^B \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

- Every concrete category of this talk is supposed to have the underlying functor $|-|$ to the respective ground category.

Varieties and their reducts

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.

- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

Definition 3

Given a variety $\mathbf{A}$, a reduct of $\mathbf{A}$ is a pair $(\| - \|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$, whereas $\mathbf{A} \xrightarrow{\| - \|} \mathbf{B}$ is a concrete functor.

Powerset and topological theories

Definition 4

A catalg backward powerset theory (cabp-theory) in a category $\mathbf{X}$ (ground category of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ to the dual category of a variety $\mathbf{A}$.

Definition 5

Let $\mathbf{X}$ be a category and let $\mathcal{T} = (P, (\| - \|, \mathbf{B}))$ comprise a cabp-theory $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ and a reduct $(\| - \|, \mathbf{B})$ of $\mathbf{A}$. A catalg topological theory (cat-theory) in $\mathbf{X}$ induced by $\mathcal{T}$ is the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, which is given by the composition $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$.

Catalg topological structures

Definition 6

Let T be a cat-theory in a category $\mathbf{X}$. $\mathbf{Top}(T)$ is the concrete category over $\mathbf{X}$, whose

objects (T-spaces) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX (T-topology on X), and whose

morphisms (T-continuous $\mathbf{X}$-morphisms) $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $(Tf)^{op}(\gamma) \in \tau$ for every $\gamma \in \sigma$.

Theorem 7

Given a cat-theory T, the category $\mathbf{Top}(T)$ is fibre-small and topological over $\mathbf{X}$.

Topological theories of O. Wyler

Definition 8

A topological theory in a category $\mathbf{X}$ is a functor $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee)$, where $\mathbf{CSLat}(\bigvee)$ is the variety of $\bigvee$-semilattices.

Definition 9

Let $\mathcal{T}$ be a topological theory in a category $\mathbf{X}$. $\mathfrak{Top}(\mathcal{T})$ is the concrete category over $\mathbf{X}$, whose

objects ($\mathcal{T}$-models) are pairs (X, t), where X is an $\mathbf{X}$-object and t is an element of $\mathcal{T}X$, and whose

morphisms ($\mathcal{T}$-morphisms) $(X, t) \xrightarrow{f} (Y, s)$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $(\mathcal{T}f)(t) \leqslant s$.

Properties of the categories $\mathfrak{Top}(\mathcal{T})$

Theorem 10

Given a topological theory $\mathcal{T}$, the category $\mathfrak{Top}(\mathcal{T})$ is fibre-small and topological over $\mathbf{X}$.

Theorem 11

For every fibre-small topological category $(\mathbf{M}, |-|)$ over $\mathbf{X}$, there exists a topological theory $\mathcal{T}$ such that $\mathbf{M}$ is concretely isomorphic to $\mathfrak{Top}(\mathcal{T})$.

Functor-costructured categories

Definition 12

Let $\mathbf{X}$ be a category and let $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}} \mathbf{Set}$ be a functor to the category $\mathbf{Set}$ of sets. $\mathbf{Spa}(\mathfrak{T})^{op}$ is the concrete category over $\mathbf{X}$, whose

objects ($\mathfrak{T}$-spaces) are pairs (X, α), where X is an $\mathbf{X}$-object and α is a subset of $\mathfrak{T}X$, and whose

morphisms ($\mathfrak{T}$-maps) $(X, \alpha) \xrightarrow{f} (Y, \beta)$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $(\mathfrak{T}f^{op})(t) \in \alpha$ for every $t \in \beta$.

Categories $\mathbf{Spa}(\mathfrak{T})^{op}$ are called functor-costructured categories.

Theorem 13

Every functor-costructured category $\mathbf{Spa}(\mathfrak{T})^{op}$ is fibre-small and topological over $\mathbf{Set}$.

Topological co-axioms

Definition 14

Let $(\mathbf{M}, |-|)$ be a concrete category over $\mathbf{X}$.

- An $\mathbf{M}$-morphism $M_1 \xrightarrow{p} M_2$ is called identity-carried provided that $|M_1| \xrightarrow{|p|} |M_2| = X \xrightarrow{1_X} X$.
- Every identity-carried $\mathbf{M}$-morphism is called a topological co-axiom in $(\mathbf{M}, |-|)$.
- An $\mathbf{M}$-object M is said to satisfy a co-axiom $M_1 \xrightarrow{p} M_2$ provided that for every $\mathbf{M}$-morphism $M \xrightarrow{f} M_2$, there exists an $\mathbf{M}$-morphism $M \xrightarrow{g} M_1$ such that $p \circ g = f$.
- A full subcategory $\mathbf{N}$ of $\mathbf{M}$ is said to be definable by topological co-axioms in $(\mathbf{M}, |-|)$ provided that there exists a class of topological co-axioms in $(\mathbf{M}, |-|)$ such that an $\mathbf{M}$-object M satisfies each of these co-axioms iff M is an $\mathbf{N}$-object.

Example of topological co-axioms

Definition 15

The contravariant powerset functor $\mathbf{Set}^{op} \xrightarrow{\mathcal{Q}} \mathbf{Set}$ is defined by $\mathcal{Q}(X \xrightarrow{f^{op}} Y) = \mathcal{P}(X) \xrightarrow{f^{\leftarrow}} \mathcal{P}(Y)$, where $f^{\leftarrow}(S) = \{y \in Y \mid f(y) \in S\}$.

Example 16

The construct **Top** of topological spaces is definable by the following (proper) class of topological co-axioms in $\mathbf{Spa}(\mathcal{Q})^{op}$:

(C_1) $(\{0\}, \{\{0\}, \varnothing\}) \to (\{0\}, \varnothing)$;

(C_2) $(\{0,1,2,3\},\{\{0,1\}, \{0,2\},\{0\}\}) \to (\{0,1,2,3\},\{\{0,1\},\{0,2\}\})$;

$(C^{\mathcal{A}}_X)$ $(X, \mathcal{A} \cup \{\bigcup \mathcal{A}\}) \to (X, \mathcal{A})$ for every set X and every family $\mathcal{A}$ of subsets of X.

Co-axiom (C_1) in more detail

Topology includes both the whole and the empty set

- Let (X, α) be a $\mathbf{Spa}(\mathcal{Q})^{op}$-object and let $X \xrightarrow{f} \{0\}$ be the unique possible map.
- Under the assumption that (X, α) satisfies co-axiom (C_1), the existence in $\mathbf{Spa}(\mathcal{Q})^{op}$ of the following triangle

$$\begin{array}{ccc} (X,\alpha) & & \\ {\scriptstyle f}\big\downarrow & \searrow^{f} & \\ (\{0\},\{\{0\},\varnothing\}) & \longrightarrow & (\{0\},\varnothing) \end{array}$$

yields $X = f^{\leftarrow}(\{0\}) \in \alpha$ and $\varnothing = f^{\leftarrow}(\varnothing) \in \alpha$.

Co-axiom (C_2) in more detail

Topology is closed under binary intersections

- Let (X,α) be a $\mathbf{Spa}(\mathcal{Q})^{op}$-object and let $S, T \in \alpha$.
- Define a map $X \xrightarrow{f} \{0,1,2,3\}$ by $f(x) = 0$, if $x \in S \cap T$; $f(x) = 1$, if $x \in S \backslash T$; $f(x) = 2$, if $x \in T \backslash S$; $f(x) = 3$, otherwise. Then $f^{\leftarrow}(\{0,1\}) = S$, $f^{\leftarrow}(\{0,2\}) = T$ and $f^{\leftarrow}(\{0\}) = S \cap T$.
- Under the assumption that (X,α) satisfies co-axiom (C_2), the existence in $\mathbf{Spa}(\mathcal{Q})^{op}$ of the following triangle

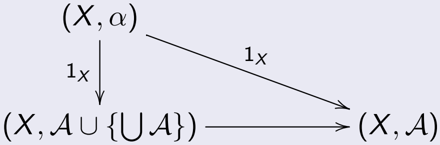
$$\begin{array}{ccc} (X,\alpha) & & \\ {\scriptstyle f}\downarrow & \searrow^{f} & \\ (\{0,1,2,3\},\{\{0,1\},\{0,2\},\{0\}\}) & \longrightarrow & (\{0,1,2,3\},\{\{0,1\},\{0,2\}\}) \end{array}$$

  yields $S \cap T = f^{\leftarrow}(\{0\}) \in \alpha$.

Co-axiom $(C_X^{\mathcal{A}})$ in more detail

Topology is closed under arbitrary unions

- Let (X, α) be a $\mathbf{Spa}(\mathcal{Q})^{op}$-object and let $\mathcal{A} \subseteq \alpha$.
- Under the assumption that (X, α) satisfies co-axiom $(C_X^{\mathcal{A}})$, the existence in $\mathbf{Spa}(\mathcal{Q})^{op}$ of the following triangle



yields $\bigcup \mathcal{A} = 1_X^{\leftarrow}(\{\bigcup \mathcal{A}\}) \in \alpha$.

- The family $(C_X^{\mathcal{A}})$ gives a proper class of topological co-axioms.

Properties of functor-costructured categories

Theorem 17

For a concrete category $(\mathbf{M}, |-|)$, *the following are equivalent:*

1. $(\mathbf{M}, |-|)$ *is fibre-small and topological;*
2. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a full concretely coreflective subcategory of some functor-costructured category;*
3. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a subcategory of some functor-costructured category* $\mathbf{Spa}(\mathfrak{T})^{op}$ *that is definable by topological co-axioms in* $\mathbf{Spa}(\mathfrak{T})^{op}$.

From Wyler to catalg

Lemma 18

There exists a functor $\mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\vdash} = A_1^d \xrightarrow{(\varphi^{\vdash})^{op}} A_2^d$, *where* $\varphi^{\vdash}$ *is the upper adjoint of* φ *in the sense of posets and* A_i^d *is the poset dual to* A_i.

Corollary 19

Every Wyler theory $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee)$ *provides the cat-theory* $\mathbf{X} \xrightarrow{T_{\mathcal{T}}} \mathbf{CSLat}(\bigvee)^{op}$, *which is defined through the composition* $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$.

$\mathfrak{Top}(\mathfrak{T})$ versus $\mathbf{Top}(T_{\mathfrak{T}})$

Theorem 20

1. *There is a full concrete embedding* $\mathfrak{Top}(\mathfrak{T}) \xhookrightarrow{F} \mathbf{Top}(T_{\mathfrak{T}})$ *defined by* $F((X,t) \xrightarrow{f} (Y,s)) = (X, \downarrow^d t) \xrightarrow{f} (Y, \downarrow^d s)$, *where* $\downarrow^d (-)$ *stands for the lower set in the dual partial order.*
2. *There is a concrete functor* $\mathbf{Top}(T_{\mathfrak{T}}) \xrightarrow{G} \mathfrak{Top}(\mathfrak{T})$ *defined by* $G((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X, \bigvee^d \tau) \xrightarrow{f} (Y, \bigvee^d \sigma)$, *where* $\bigvee^d$ *stands for the join in the dual partial order.*
3. G *is a right-adjoint-left-inverse to* F.

Corollary 21

$\mathfrak{Top}(\mathfrak{T})$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Top}(T_{\mathfrak{T}})$.

Properties of catalg topology

Proposition 22

Given a cat-theory T, every full concretely coreflective subcategory $(\mathbf{M}, |-|)$ of the category $\mathbf{Top}(T)$ is finally closed in $\mathbf{Top}(T)$.

Proposition 23

Given a cat-theory T, for every concrete category $(\mathbf{M}, |-|)$, the following are equivalent:

1. $\mathbf{M}$ *is a full concretely coreflective subcategory of* $\mathbf{Top}(T)$;
2. $\mathbf{M}$ *is definable by topological co-axioms in* $\mathbf{Top}(T)$.

Catalg topology versus categorical topology

Theorem 24

For a concrete category $(\mathbf{M}, |-|)$, *the following are equivalent:*

1. $(\mathbf{M}, |-|)$ *is fibre-small and topological;*
2. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a subcategory of a category* $\mathbf{Top}(T)$ *that is definable by topological co-axioms in* $\mathbf{Top}(T)$.

Proof.

(1) $\Rightarrow$ (2): There is a Wyler theory $\mathcal{T}$ such that $\mathbf{M}$ is concretely isomorphic to $\mathfrak{Top}(\mathcal{T})$, which is concretely isomorphic to a full concretely coreflective subcategory of $\mathbf{Top}(T_{\mathcal{T}})$, i.e., $\mathbf{M}$ is concretely isomorphic to a full concretely coreflective subcategory of $\mathbf{Top}(T_{\mathcal{T}})$.

(2) $\Rightarrow$ (1): $\mathbf{M}$ is concretely isomorphic to a full concretely coreflective subcategory of $\mathbf{Top}(T)$, i.e., is fibre-small and finally closed in $\mathbf{Top}(T)$. Since $\mathbf{Top}(T)$ is topological, $\mathbf{M}$ must be as well.

From catalg to Wyler

Lemma 25

Given a variety $\mathbf{A}$*, there exists a functor* $\mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\!\!\!\!\!\!\!\!\circ}} \mathbf{CSLat}(\bigvee)$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\leftarrow\!\!\!\!\!\!\!\!\circ} = (Sub(A_1))^d \xrightarrow{(\varphi^{op})^{\leftarrow}} (Sub(A_2))^d$*, where* $Sub(A_i)$ *is the* $\bigwedge$*-semilattice of subalgebras of* A_i*, whereas* $(\varphi^{op})^{\leftarrow}(S) = \{a \in A_2 \,|\, \varphi^{op}(a) \in S\}$.

Corollary 26

Every cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ *provides a Wyler theory* $\mathbf{X} \xrightarrow{\mathcal{T}_T} \mathbf{CSLat}(\bigvee)$ *defined by the composition* $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\!\!\!\!\!\!\!\!\circ}} \mathbf{CSLat}(\bigvee)$.

Theorem 27

The categories $\mathbf{Top}(T)$ *and* $\mathfrak{Top}(\mathcal{T}_T)$ *are equal.*

From functor-costructured to catalg

Remark 28

Given a functor $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}} \mathbf{Set}$, *there exists the functor* $\mathbf{X} \xrightarrow{T_{\mathfrak{T}}} \mathbf{Set}^{op}$ *defined as* $\mathbf{X} \xrightarrow{\mathfrak{T}^{op}} \mathbf{Set}^{op}$.

Theorem 29

The categories $\mathbf{Spa}(\mathfrak{T})^{op}$ *and* $\mathbf{Top}(T_{\mathfrak{T}})$ *are equal.*

From catalg to functor-costructured

Remark 30

Every cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ *gives rise to the functor* $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}_T} \mathbf{Set}$, *which is defined through the composition* $\mathbf{X}^{op} \xrightarrow{T^{op}} \mathbf{A} \xrightarrow{|-|} \mathbf{Set}$, *where* $|-|$ *is the underlying functor of the variety* $\mathbf{A}$.

$\mathbf{Top}(T)$ versus $\mathbf{Spa}(\mathfrak{T}_T)^{op}$

Theorem 31

1. *There is a full concrete embedding* $\mathbf{Top}(T) \xhookrightarrow{F} \mathbf{Spa}(\mathfrak{T}_T)^{op}$ *defined by* $F((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X,|\tau|) \xrightarrow{f} (Y,|\sigma|)$.
2. *There is a concrete functor* $\mathbf{Spa}(\mathfrak{T}_T)^{op} \xrightarrow{G} \mathbf{Top}(T)$ *defined by* $G((X,\alpha) \xrightarrow{f} (Y,\beta)) = (X,\langle\alpha\rangle) \xrightarrow{f} (Y,\langle\beta\rangle)$, *where* $\langle S\rangle$ *stands for the subalgebra generated by a set* S.
3. G *is a right-adjoint-left-inverse to* F.

Corollary 32

$\mathbf{Top}(T)$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Spa}(\mathfrak{T}_T)^{op}$.

Final remarks

- Following the rapid development of both catalg topology and categorical topology, this talk clarified the relationships between these two approaches to the study of topological structures.
- The setting of topological theories of O. Wyler is more general than the catalg one, in the sense that every category of the form $\mathbf{Top}(T)$ can be reconstructed completely through a suitable category of the form $\mathfrak{Top}(\mathcal{T})$, whereas the converse way requires the application of some topological co-axioms, whose ultimate description in each case can be problematic.
- In concrete applications, catalg framework appears to be more suitable, since it provides the underlying algebraic structures of the topological structures, whereas topological theories of O. Wyler contain the information on their ground category only.

References I

- J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications, 2009.
- J. Adámek, J. Rosický, E. M. Vitale, *Algebraic Theories. A Categorical Introduction to General Algebra*, Cambridge University Press, 2011.
- H. Herrlich, *Categorical topology*, General Topology Appl. **1** (1971) 1–15.
- H. Herrlich, *Universal Topology*, in: H. L. Bentley, H. Herllich, M. Rajagopalan, H. Wolff (eds.), Categorical topology, Proc. Int. Conf., Toledo/Ohio 1983, Heldermann Verlag, 1984, pp. 223–281.
- S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, Int. J. Math. Math. Sci. **2007** (2007) 1–71.

References II

- S. Solovjovs, *Categorically-algebraic topology*, in: Abstracts of the International Conference on Algebras and Lattices, Charles University, Prague, 2010, pp. 20–22.
- S. Solovyov, *Fuzzy algebras as a framework for fuzzy topology*, Fuzzy Sets Syst. **173** (1) (2011) 81–99.
- S. Solovyov, *Generalized fuzzy topology versus non-commutative topology*, Fuzzy Sets Syst. **173** (1) (2011) 100–115.
- O. Wyler, *On the categories of general topology and topological algebra*, Arch. Math. **22** (1971) 7–17.
- O. Wyler, *TOP categories and categorical topology*, General Topology Appl. **1** (1971) 17–28.

Thank you for your attention!